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THE CLASSIFICATION OF DISCRETE 2-GENERATOR SUBGROUPS OF PSL(2, R)

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ABSTRACT

This paper gives a short geometric algorithm for deciding the discreteness of most 2-generator subgroups of PSL(2, **R**), as well as a self-contained algorithmic approach to the complete classification.

Our goal is to give a new and self-contained proof of the classification of 2-generator Fuchsian groups. This problem has been discussed in part by Knapp [1] and then extensively by Purzitsky [5–9]. If $A, B \in \mathrm{PSL}(2, \mathbf{R})$, neither the identity, then there are seven cases to consider:

	A	B	
1.	elliptic	elliptic	
2.	elliptic	parabolic	
3.	parabolic	parabolic	
4.	elliptic	hyperbolic	
5.	parabolic	hyperbolic	
6.	hyperbolic	hyperbolic	axes disjoint
7.	hyperbolic	hyperbolic	axes intersecting

This numbering of the cases will be used throughout this paper. In Case 1, Knapp's result decides the discreteness of $G = \langle A, B \rangle$. We show that deciding discreteness in Cases 2–6 reduces immediately to previous cases via a short algorithm. Thus Cases 1–6 are strongly unified. Our technique is to adjoin to G a particular reflection of $\mathbf{H}^2$, related to the Lie product of A and B, which gives strong geometric information about G and a nearly canonical fundamental domain when G is discrete. Case 7, on the other hand, is qualitatively different and requires a long algorithm. We point out and correct an error in the

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treatment of Case 7 in [9]. In conclusion, we give a new proof of the existence of uniform collars (with sharp constants) used in [4].

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The Classification

We think of A and B as acting as isometries of $\mathbf{H}^2$ which preserve orientation. The unit disc will be our standard model of $\mathbf{H}^2$. An elliptic, parabolic, or hyperbolic isometry of $\mathbf{H}^2$ is respectively a rotation, limit rotation, or translation. In Cases 1–6 we make the following normalizations: A and B either fix a point of the real axis or have axis of translation perpendicular to the real axis, A being to the left of the imaginary axis and B to the right — these adjustments can be accomplished by changing coordinates. Then, by replacing A by A^{-1} or B by B^{-1} if need be, we assume that A and B map 0 to the (closed) upper half plane. For convenience, call $X \in \mathrm{PSL}(2,\mathbf{R})$ *primitive* if X is hyperbolic, parabolic, or elliptic of order l (written $l = o(X)$) and X rotates by $2\pi/l$.

THEOREM. *If $A, B \in \mathrm{PSL}(2,\mathbf{R})$ are primitive and normalized as above, then $G = \langle A, B\rangle$ is discrete if and only if the following conditions of an algorithmically decidable nature are satisfied:*

Case 1 (*Knapp* [1]): *AB^{-1} is primitive or one of the following five conditions holds: set $n = o(A)$, $m = o(B)$, $n \leqq m$, also set $l = o(AB^{-1})$ and let AB^{-1} rotate by $k(2\pi/l)$ so that k and l are relatively prime.*

(i) $k = 2,\ n = m,\ 1/l + 1/n < 1/2$;
(ii) $k = 2,\ n = 2,\ m = l,\ m \geqq 7$;
(iii) $k = 3,\ n = 3,\ m = l,\ m \geqq 7$;
(iv) $k = 4,\ n = m = l,\ m \geqq 7$;
(v) $k = 2,\ n = 3,\ m = l = 7$.

Case 2: *AB^{-1} is primitive.*

Case 3: *AB^{-1} is primitive, or AB^{-1} rotates by $2(2\pi/l)$ where $o(AB)^{-1} = l$.*

Cases 4, 5, *and* 6: *A^NB^{-1} is not elliptic for all $N \geqq 1$, or if $A^{N_0}B^{-1}$ is elliptic for some $N_0 \geqq 1$, $\langle A^{N_0}B^{-1}, A\rangle$ is discrete (which reduces to a previous case).*

Case 7 (*Purzitsky* [9]): *$ABA^{-1}B^{-1}$ is primitive or the square of a primitive element, or A and B, up to a conjugation, occur in one of three well-known discrete* *groups, a* $(2,3,\nu)$ *or a* $(2,4,\nu)$ *or a* $(3,3,\nu)$ *triangle group, where* $\nu = o(ABA^{-1}B^{-1})$.

PROOF.

Case 1: Let A, B fix $a, b \in \mathbf{H}^2$ respectively, $o(A) = n$, $o(B) = m$. Construct the convex, three-sided polygon T in $\mathbf{H}^2$ with vertices at a and b and with angles π/n at a and π/m at b. Let R, R_a, R_b be the reflections in the sides of T where R fixes a and b. Then $A = R_aR$, $B = R_bR$. Note that $AB^{-1} = R_aRRR_b = R_aR_b$ measures the relative positions of the axes of R_a and R_b.

Let $\tilde{G} = \langle A, B, R\rangle$; then $G = \langle A, B\rangle$ is the orientation preserving subgroup of $\tilde{G}$ of index 2 and G and $\tilde{G}$ are discrete or non-discrete together.

If AB^{-1} is primitive, i.e., AB^{-1} is hyperbolic, parabolic, or elliptic rotating by $2\pi/o(AB^{-1})$, then T satisfies the hypotheses of Poincaré's Polygon Theorem, see Maskit [3], and we may conclude that $\tilde{G}$ is discrete with T as fundamental domain. A fundamental domain for G can be obtained by juxtaposing T and RT.

Suppose that T has a third vertex c so that AB^{-1} is elliptic. If the angle of T at c is irrational times π, then $\tilde{G}$ is not discrete. If the angle at c is $k(\pi/l)$, $k > 1$, $l = o(AB^{-1})$, Knapp's Theorem says that there are just five situations in which $\tilde{G}$ is discrete. We now give a proof of this result designed to exploit the reflections in $\tilde{G}$.

Since $k > 1$, there are additional axes of reflection in $\tilde{G}$ passing through c and making angles of $\pi/l, 2(\pi/l), \cdots, (k-1)(\pi/l)$ with the segment ac. We say that these reflections are implied by the angle at c. Consider the 2 end triangles, $\Delta ad'c$ and Δcdb, so formed in T and call these the first generation ends. Repeat this procedure with $\Delta ad'c$ and with Δcdb to obtain, at most, 4 triangles called the second generation ends. There are, similarly, at most 2^N N-th generation ends. We can decide the discreteness of $\tilde{G}$ by induction on N such that the process terminates with the N-th generation ends. The process must indeed terminate if $\tilde{G}$ is discrete because only finitely many axes of reflections can cross T.

Assume that we have termination with $N = 1$; then $\sphericalangle ad'c$ and $\sphericalangle cdb$ must both be $\leqq \pi/2$ because an obtuse angle always implies more reflections. Thus $d = d'$, cd is an altitude of Δabc, $k = 2$, and so $n = m$ and (i) holds.

If we have termination with $N = 2$, either $\Delta ad'c$ or Δcdb must be obtuse, say Δcdb is, for otherwise (i) would hold for Δabc. Note that Δcdb must terminate after one step, so the above implies that $\sphericalangle cdb = 2\pi/\alpha$, α an integer. Since $2\pi/\alpha > \pi/2$, we conclude that $\alpha = 3$. If $k = 2$, then (ii) holds. If $k = 3$, (iii) holds.

If $k \geqq 4$, then (iv) holds because we must have that $\Delta cd'a$ is obtuse and so $\sphericalangle cd'a = 2\pi/3$ and $k = 4$ follows.

Assume that the process stops with $N = 3$. We may take Δcdb to be obtuse. Δcdb then terminates on step 1 or 2. If (i) holds for Δcdb, then $\sphericalangle cdb = 2\pi/3$ because this is the only obtuse angle with numerator 2 and odd denominator. If (ii) holds for Δcdb, then $\sphericalangle cdb = 2\pi/3$ again and we are forced to have a subtriangle with angles $\pi/2$, $\pi/3$, $\pi/3$ which is impossible in hyperbolic geometry. Similarly, if (iii) holds for Δcdb, then $\sphericalangle cdb = 3\pi/4$ or $3\pi/5$ leading to an impossible triangle. If (iv) holds, then $\sphericalangle cdb = 4\pi/7$. Thus a 1 or 2 step obtuse end triangle has obtuse angle $2\pi/3$ and (i) holds or $4\pi/7$ and (iv) holds. Now if $k = 2$, then Δabc satisfies (v). We can rule out $k > 2$ as follows: $\sphericalangle cd'a$ would then be $\geqq 2\pi/3$ because the configuration (v) sits inside Δabc. The only way for $\Delta cd'a$ to terminate in 1 or 2 steps is for $k = 3$, (i) holds, and $\sphericalangle cd'a = 2\pi/3$ — this readily gives the contradiction $\pi/2 = 3\pi/7$ for instance. So there is a unique 3 step triangle.

Lastly, assume that the process terminates with $N = 4$. Then Δcdb say is a 3 step triangle and $\sphericalangle cdb = 2\pi/7$. This forces $\sphericalangle cd'a \geqq 5\pi/7$ which cannot be in a 1 or 2 step triangle. Hence there are no 4 step triangles and no N step triangles with $N \geqq 4$.

Case 2: Construct a three sided convex polygon T with angles π/n at a and 0 at b where A fixes a, $o(A) = n$, and B fixes b. If R, R_a, and R_b are the reflections in the sides of T as above, then $A = R_aR$, and $B = R_bR$.

If AB^{-1} is primitive, then $\tilde{G}$ is discrete with T as fundamental domain. The converse also holds — let R_c be a further reflection implied by a third vertex c of T. Say that the axis of R_c meets ab at d. Then Δacd contains a fundamental domain of $\tilde{G}$. Since the cusped triangle Δcdb has finite area, it must be tiled by finitely many copies of the fundamental domain, but Δcdb is not compact — contradiction.

Case 3: Construct T again so that $A = R_aR$, $B = R_bR$, A fixes a, and B fixes b, $a \neq b$. If AB^{-1} is primitive, we are done. If $o(AB^{-1}) = l$ and AB^{-1} rotates by $k(2\pi/l)$ about c, $k > 1$, then $k = 2$; if $k > 2$, there would be a compact fundamental domain; but, as in Case 2, we could not then tile the cusps. Since $k = 2$, exactly one half of T is a fundamental domain for $\tilde{G}$ — T behaves like an isoceles triangle.

Case 4: Let R be the reflection with axis passing through a, the fixed point of A, and perpendicular to the axis of B. a might well lie on the axis of B. Consider the factorizations $A = R_aR$, $B = R_bR$ where R_a and R_b are reflections as before. Also write $A^N = R_{aN}R$. If $A^NB^{-1} = R_{aN}R_b$ is never elliptic, then $\tilde{G}$ is discrete: there is exactly one N_0 mod n, $n = o(A)$, such that the axis of R_b is contained in the sector bounded by the axes of R_{aN_0} and $R_{a(N_0+1)}$. The convex set bounded by the axes of R_{aN_0}, $R_{a(N_0+1)}$, and R_b is then a fundamental domain for $\tilde{G}$.

If $A^{N_0}B^{-1} = R_{aN_0}R_b$ is elliptic for some N_0, then we apply Case 1 to the generators $A^{N_0}B^{-1}$ and A to decide discreteness. For clarity, we carry this out. Let T be the convex set bounded by the axes of R_{aN_0}, $R_{a(N_0+1)}$, and R_c, where R_c is a reflection whose axis makes an angle of π/m, $m = o(A^{N_0}B^{-1})$, with the axis of R_{aN_0} at c, the fixed point of $A^{N_0}B^{-1}$. Either T is a fundamental domain for $\tilde{G}$, or T is one of Knapp's five triangles, or $\tilde{G}$ is not discrete.

Case 5: Note that if A and B share a fixed point, then G is not discrete. (This is well known, but can be proved by adjoining the reflection R' in the axis of B — consideration of the images of the axis of the reflection AR' under powers of B shows nondiscreteness.) Otherwise, proceed as in Case 4, using Case 2 to finish the argument.

Case 6: Again, if A and B share just one fixed point G is not discrete. (Look at the subgroup $H = \langle A, BAB^{-1}\rangle$ of G and observe that A and BAB^{-1} share just one fixed point; there is a reflection R' such that $R'AR' = BAB^{-1}$; $\tilde{H} = \langle H, R'\rangle$ is non-discrete as before.) Otherwise, let R be the reflection in the common perpendicular to the axes of A and B and proceed as in Case 4 using Case 4 a second time to finish the argument.

Case 7: Consider A, B hyperbolic with axes intersecting in exactly one point p. Let E be elliptic of order 2 fixing p. E was introduced by Purzitsky in [9]. Observe that $EAE = A^{-1}$ and $EBE = B^{-1}$. Set $A = E_aE$ and $B = E_bE$ so that E_a and E_b are elliptic of order 2. Note also that $ABA^{-1}B^{-1} = ABEAEB^{-1} = AE_bAE_b = (AE_b)^2$. AE_b is a square root of the commutator. It is easy to construct AE_b: drop a perpendicular l from the fixed point of E_b to the axis of A and do the same for AE_bA^{-1} to obtain l'. Draw perpendiculars λ and λ' to l and l' through the fixed points of E_b and AE_bA^{-1} respectively. AE_b then maps λ to λ' taking the fixed point of AE_bA^{-1} to the fixed point of E_b. So, for instance, AE_b is elliptic if and only if λ and λ' intersect and then AE_b fixes the point of intersection.

Let $G' = \langle A, B, E\rangle$; G' has $G = \langle A, B\rangle$ as a subgroup of index 2 or $G = G'$. Now the convex polygon P bounded by the axis of A and by l, l', λ, and λ', is a fundamental domain of G' if AE_b is primitive.

Assume that $o(AE_b) = n$ and that AE_b rotates by $k(2\pi/n)$, $k > 1$. To deal with this situation, we use the *area formula* for Fuchsian groups, see [2], p. 77.

If Γ is a finitely generated Fuchsian group with area $(\mathbf{H}^2/\Gamma) < \infty$, then

$$\operatorname{area}(\mathbf{H}^2/\Gamma) = 2\pi \left\{ 2(g-1) + \sum_{j=1}^{m} \left(1 - \frac{1}{\nu_j}\right) \right\}$$

where $g = \operatorname{genus}(\mathbf{H}^2/\Gamma)$ and Γ has m conjugacy classes of maximal elliptic and parabolic cyclic subgroups of orders ν_j, $2 \leqq \nu_j \leqq \infty$, $\nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_m$.

Note that $\operatorname{area}(P) = \pi - 2\pi k/n$. Since the sides of P are identified by elements of G', it follows that P contains a fundamental domain of G' provided G' is discrete. In fact, $\operatorname{area}(P) = p \cdot \operatorname{area}(\mathbf{H}^2/G')$ for some integer p, as one readily sees.

The inequality area $(\mathbf{H}^2/G') < \operatorname{area}(P)$ implies that $g = 0$ and $m = 3$, so that G' must be a triangle group.

The inequality $2 \cdot \operatorname{area}(\mathbf{H}^2/G') \leqq \operatorname{area}(P)$ can be written

$$\frac{k}{2n} \leqq \left(\frac{1}{\nu_1} + \frac{1}{\nu_2} - \frac{3}{4}\right) + \frac{1}{\nu_3}\ .$$

If $(\quad) \leqq 0$, then $k/2 \leqq n/\nu_3$, but $k \geqq 2$ and $n \leqq \nu_3$; therefore $k = 2$, $n = \nu_3$, $(\quad) = 0$, $\nu_1 = 2$, $\nu_2 = 4$ and $p = 2$, see Fig. 1. If $(\quad) > 0$, then $\nu_1 = 2$ and $\nu_2 = 3$. In this case $p \cdot \operatorname{area}(\mathbf{H}^2/G') = \operatorname{area}(P)$ can be written $(p/3)(\nu_3 - 6) = \nu_3 - 2dk$ where $d = \nu_3/n$ is an integer. Now if $p = 2$, then as one checks by counting the possible number of angles of $2\pi/\nu_3$ that can occur in P, $k = 2$ and $d = 1$, and there are no solutions. If $p \geqq 3$, then $\nu_3 - 6 \leqq \nu_3 - 2dk$, or $dk \leqq 3$, so that $d = 1$ and $k = 2, 3$. If $k = 2$, then we are led to $p = 9$ and $\nu_3 = 7$ which actually occurs (see Fig. 1), or to $p = 5$ and $\nu_3 = 9$ which cannot occur in our situation. If $k = 3$, then $p = 3$; again see Fig. 1.

We have shown that if G' is discrete, then G' is a $(2,3,n)$ or a $(2,4,n)$ triangle group where $n = o(AE_b)$. If $k = 3$ we have the first possibility and if $k = 2$ we have the second possibility unless $n = 7$ when we have both possibilities — this last fact was omitted in Purzitsky [9].

We claim that $G = G'$ if and only if n is odd and that G' has index 2 in G' if and only if n is even, $n = o(AE_b)$. Set $\nu = o(ABA^{-1}B^{-1})$. If n is even, then $n = 2\nu$ and G' is a $(2,3,n)$ triangle group. A $(2,3,n)$ triangle group has a subgroup of index 2 precisely when n is even, this subgroup being a $(3,3,\nu)$ triangle group. If n is odd, then $n = \nu$ because $ABA^{-1}B^{-1} = (AE_b)^2$ and thus $G = G'$. To complete the argument, assume that n is even and observe that E, or E_a, or E_b, can not be an element of the $(3,3,\nu)$ subgroup: if ν is odd, this is

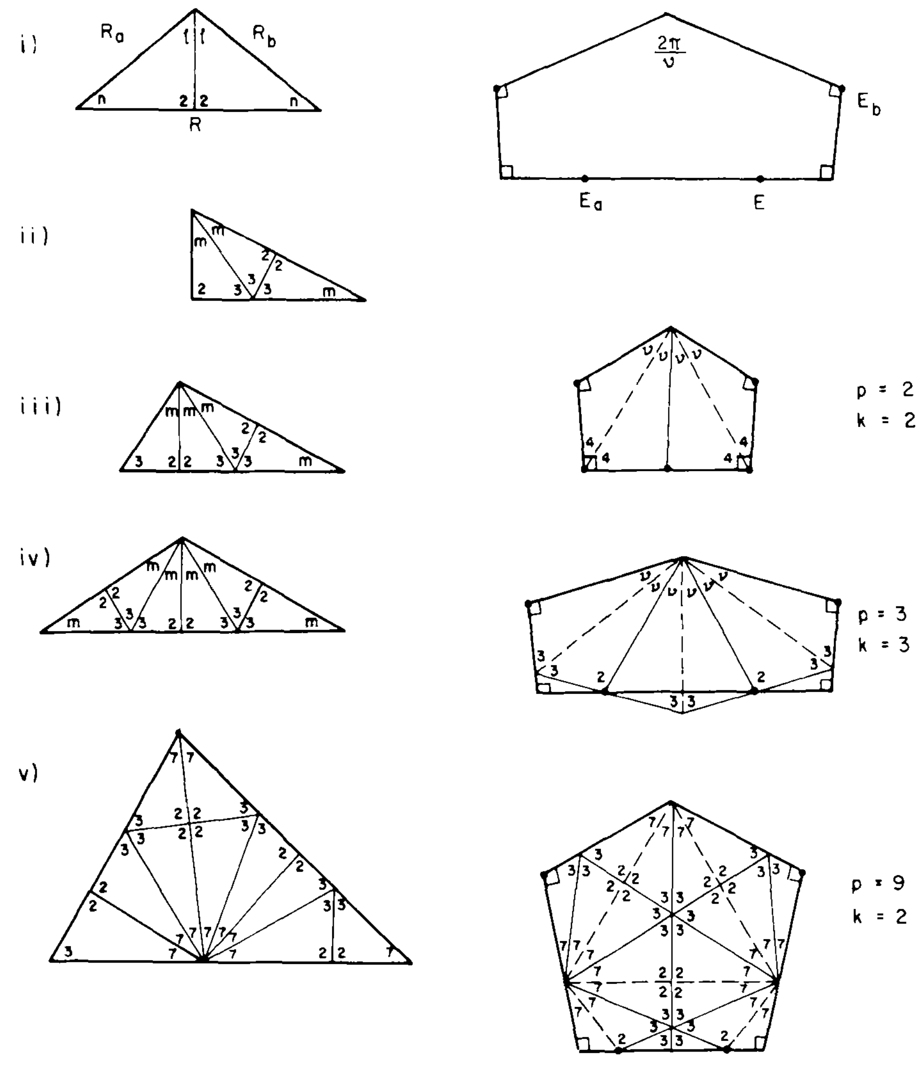

Fig. 1.

clear, otherwise, look at P and note that there would be too many angles of $2\pi/n$ to be consistent with $p = 3$. So A or $AE = E_a$ and B or $BE = E_b$ are in the subgroup and therefore G has index 2 in G'.

In Fig. 1, the model situations for G' discrete are shown. In each of these examples, one can repeatedly do "leap frog" operations on the generators E, E_a, and E_b and still obtain the same group. Purzitsky has shown that any set of three elliptic elements of order 2 generating a discrete group is Nielsen equivalent to one of the standard sets. Another test for discreteness is to determine when E, E_a, and E_b, up to a conjugation, occur in a $(2,3,n)$ or a $(2,4,n)$ triangle group — this can clearly be accomplished in a finite number of steps.

Collars

Let Γ be a Fuchsian group. The axis of a hyperbolic element of Γ is called *simple* if it projects to a simple closed geodesic on $\mathbf{H}^2/\Gamma$, i.e., if it is precisely invariant under its stabilizer in Γ which is hyperbolic cyclic. A *collar* about a simple axis is a neighborhood which we take to be of constant width which is also precisely invariant under the stabilizer of the simple axis.

Say that $A \in \Gamma$ generates the stabilizer of a simple axis. For any $X \in \Gamma - \langle A \rangle$, the axis of XAX^{-1} is disjoint from the axis of A and the axis of A is simple with respect to $\langle A, XAX^{-1}\rangle$. Now $\langle A, XAX^{-1}\rangle$ is a subgroup of $\langle A, E'\rangle$ of index 1 or 2, where E' is elliptic of order 2 fixing the midpoint of the segment perpendicular to the axis of A and the axis of XAX^{-1}; the axis of A remains simple with respect to $\langle A, E'\rangle$. We ask the following question: how close can the fixed point of an elliptic element E of order 2 be to the axis of A such that $\langle A, E\rangle$ is discrete and the axis of A is simple in $\langle A, E\rangle$? This can be resolved using our present methods.

Consider $\langle A, E, R\rangle$ where R is the reflection with axis passing through the fixed point of E and perpendicular to the axis of A. A and E factor into $A = R_aR$, $E = R_eR$, where R_a and R_e are reflections, the axis of R_e being perpendicular to the axis of R. If $AE = R_aR_e$ is hyperbolic, parabolic, or primitive elliptic, then $\langle A, E\rangle$ is discrete and the axis of A is simple in $\langle A, E\rangle$. The converse also holds: if AE is elliptic but not primitive, then suppose that $\langle A, E\rangle$ is discrete and construct a triangular fundamental domain for $\langle A, E\rangle$ which intersects the axis of A, is invariant under R and which is bounded by the axes of reflections in $\langle A, E, R\rangle$. One readily sees that the axis of A is not simple with respect to $\langle A, E\rangle$. We conclude that E_0 is closest satisfying the conditions exactly when AE_0 is elliptic of order 3, and the distance from the axis of A to the fixed point of E_0 is the width of the best possible uniform collar.

Similar but easier reasoning can be carried out when A is parabolic or elliptic. If A is parabolic or elliptic, $o(A) \geqq 7$, the answer again is $o(AE) = 3$. If $o(A) = 6$ or 5, the answer is $o(AE) = 4$. If $o(A) = 4$, then $o(AE) = 5$ and if $o(A) = 3$, $o(AE) = 7$. One can use hyperbolic trigonometry to find the widths of the various collars and to show that they are mutually disjoint on $\mathbf{H}^2/\Gamma$ as in [4].

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